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A minimax theorem for infinite graphs with ideal points

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Abstract

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Let $\mathscr{A}$ be a family of sets of ends of an infinite graph, having the property that every element of any member of $\mathscr{A}$ can be separated from the union of all other members by a finite set of vertices. By defining appropriate concepts of $\mathscr{A}$-paths and of $\mathscr{A}$-separators, we show that there are a set of pairwise disjoint $\mathscr{A}$-paths and an $\mathscr{A}$-separator which have the same 'cardinality'.

Introduction

In a recent paper [8] we generalized different extensions [10, 3, 5] of Menger's Theorem to infinite graphs admitting their ends as ideal points. We proved, by defining appropriate concepts of AB-paths and of AB-separators, that, if A and B are two sets of ends of an infinite graph G, having the property that any element of A (resp. B) can be separated from B (resp. A) by a finite set of vertices, then there are a set of pairwise disjoint AB-paths and an AB-separator which have the same cardinality.

In this result, as in any Menger-like theorem, the separation of exactly two sets of objects of a graph is involved. A more interesting and much more general result would be a similar theorem dealing with any set $\mathscr{A} := \{A_i : i \in I\}$ of pairwise disjoint sets of ends of a graph G. This is this generalization which is studied in this paper.

For such a set $\mathscr{A} = \{A_i : i \in I\}$ we define an *$\mathscr{A}$-path* as an A_iA_j-path (as defined in [8]) for any different $i, j \in I$, and an *$\mathscr{A}$-separator* as an ordered pair (S, T) with $S \subseteq V(G)$ and $T \subseteq E(G)$ such that any $\mathscr{A}$-path has a vertex in S or an edge in T, and whose delection does not split or destroy any end in $\bigcup \mathscr{A}$. The existence of an $\mathscr{A}$-separator is ensured when, for any $i \in I$, every element of A_i can be

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separated from $\bigcup_{j\neq i} A_j$ by a finite set of vertices. Then, using the extension to infinite graphs of a theorem of Mader [4] that we give in Section 1, we get a minimax theorem between the cardinalities of the pairwise disjoint sets of $\mathscr{A}$-paths and those—actually a modified version of cardinality—of the $\mathscr{A}$-separators, for any set $\mathscr{A}$ having the preceding property, so generalizing Theorem 3.3 of [8].

0. Preliminaries

0.1. If X is a set we denote by $|X|$ its cardinality, by $\mathscr{P}(X)$ its power set, and if n is a cardinal, by $[X]^n$ (resp. $[X]^{<n}$, $[X]^{\geqslant n}$) the set of its subsets of cardinality n (resp. $<n$, $\geqslant n$). Furthermore, $\lfloor |X|/2 \rfloor$ will be $\max\{n \in \mathbb{N}: n \leqslant |X|/2\}$ if X is finite, and $|X|$ otherwise.

0.2. A *graph* G is a set $V(G)$ (*vertex set*) together with a set $E(G) \subseteq [V(G)]^2$ (*edge set*). H is a *subgraph* of G ($H \subseteq G$) if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is an *induced subgraph* if $E(H) = [V(H)]^2 \cap E(G)$. For $A \subseteq V(G)$ we denote by $G \mid A$ the subgraph induced by A. If B is any set, and H any graph, we define $G - B := G \mid (V(G) - B)$ and $G - H := G - V(H)$. For $T \subseteq E(G)$ we denote by $G(T)$ the subgraph of G whose edge set is T, and vertex set is $\bigcup T$. The *union* of a family $(G_i)_{i\in I}$ of graphs is the graph $G := \bigcup_{i\in I} G_i$ given by $V(G) = \bigcup_{i\in I} V(G_i)$ and $E(G) = \bigcup_{i\in I} E(G_i)$. The *intersection* is defined similarly. If X and Y are subgraphs of G, then the *boundary of X with Y* is the set $\mathscr{B}(X, Y) := \{x \in V(X): \{x, y\} \in E(G)$ for some $y \in V(Y - X)\}$. In particular $\mathscr{B}(X) := \mathscr{B}(X, G)$ is the *boundary* of X. The set of components of G is denoted by $\mathscr{C}_G$, and if x is a vertex, then $\mathscr{C}_G(x)$ is the component of G containing x. A *path* $W := \langle x_0, \ldots, x_n \rangle$ is a graph with $V(W) = \{x_0, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(W) = \{\{x_i, x_{i+1}\}: 0 \leqslant i < n\}$. A *ray* or *one-way infinite path* $R := \langle x_0, x_1, \ldots \rangle$, and a *double ray* or *two-way infinite path* $D := \langle \ldots, x_{-1}, x_0, x_1, \ldots \rangle$ are defined similarly. A path $\langle x_0, \ldots, x_n \rangle$ is called an *x_0x_n-path*. For $A, B \subseteq V(G)$, an *AB-path* of G, or simply an *A-path* when $A = B$, is an xy-path of G whose only vertices in $A \cup B$ are x and y, with $x \in A$ and $y \in B$. Two paths are *internally disjoint* if they have at most their endpoints in common.

0.3. The *ends* of a graph G (this concept was introduced by Freudenthal [1] and independently by Halin [2]) are the classes of the equivalence relation $\sim_G$ defined on the set of all rays of G by $R \sim_G R'$ if and only if there is a ray R'' whose intersections with R and R' are infinite; or equivalently if and only if $\mathscr{C}_{G-S}(R) = \mathscr{C}_{G-S}(R')$ for any $S \in [V(G)]^{<\omega}$ (where $\mathscr{C}_{G-S}(R)$ denotes the component of $G - S$ containing a subray of R). We will use the following notations:

- if R is a ray of G, then $[R]_G$ is the class of R modulo $\sim_G$;
- $\mathfrak{T}(G)$ is the set of all ends of G;
- if H is a subgraph of G, then $\mathfrak{T}_H(G)$ is the set of ends of G having rays of H as elements;
- if $\tau \in \mathfrak{T}(G)$ and $S \subseteq V(G)$, then $\mathscr{C}_{G-S}(\tau)$ is the component of $G - S$ containing a ray in τ; and if $A \subseteq \mathfrak{T}(G)$ then $\mathscr{C}_{G-S}(A) := \bigcup_{\tau \in A} \mathscr{C}_{G-S}(\tau)$;
- if $\tau \in \mathfrak{T}(G)$ then $V_\tau := \{x \in V(G): x \in V(\mathscr{C}_{G-S}(\tau))$ for any $S \in [V(G) - \{x\}]^{<\omega}\}$.

0.4. For every $S \in [V(G)]^{<\omega}$ we denote by S^* the equivalence relation on $\mathfrak{T}(G)$ defined by $(\tau, \tau') \in S^*$ if and only if $\mathscr{C}_{G-S}(\tau) = \mathscr{C}_{G-S}(\tau')$. The family of all such equivalence relations is a base of a Hausdorff uniformity on $\mathfrak{T}(G)$ (see [5]). We will assume that, throughout this paper, the end set $\mathfrak{T}(G)$ of a graph G is endowed with the topology induced by this uniformity, i.e., the topology such that, for every end τ, the family $(S^*(\tau))_{S \in [V(G)]^{<\omega}}$ is a local basis at the point τ. For any $A \subseteq \mathfrak{T}(G)$ we will denote by $\bar{A}$ the closure of A for this topology.

0.5. An infinite subset S of $V(G)$ is *concentrated* in G if it has the following equivalent properties [5, Theorem 3.5]:

(i) there is an end τ such that $S - V(\mathscr{C}_{G-F}(\tau))$ is finite for any $F \in [V(G)]^{<\omega}$ (S is said to be *concentrated in* τ);

(ii) for all $T, U \in [S]^{\geq\omega}$, there is a family of pairwise TU-paths in G.

0.6. *Let* $A \subseteq \mathfrak{T}(G)$. A subset S of $V(G)$ is A-*dispersed* if it has the following equivalent properties [8, Proposition 0.6]:

(i) for every $\tau \in A$ there is an $F \in [V(G)]^{<\omega}$ such that $S \cap V(\mathscr{C}_{G-F}(\tau)) = \emptyset$;

(ii) S has no subset concentrated in some end $\tau \in A$.

We denote by $D(A)$ the set of A-dispersed subsets of $V(G)$. $S \in D(A)$ is *finitary* if, for any $\tau \in A$, there is a finite subset F of S such that $\mathscr{C}_{G-F}(\tau) = \mathscr{C}_{G-S}(\tau)$. By Proposition 0.13 of [8], for any $S \in D(A)$ there is a finitary $T \in D(A)$ which contains S.

1. An extension of a theorem of Mader to infinite graphs

1.1 Let A be an independent set of vertices of a graph G (i.e., $E(G) \cap [A]^2 = \emptyset$). An ordered pair (S, T) is called an A-*separator* of G if it has the following properties:

(i) $S \subseteq V(G) - A$;

(ii) $T \subseteq E(G) \cap [V(G) - A \cup S]^2$;

(iii) for every A-path W, $V(W) \cap S \neq \emptyset$ or $E(W) \cap T \neq \emptyset$.

We will denote by $\mathrm{DISJ}_G(A)$ the set of all sets of pairwise internally disjoint A-paths of G, and by $\mathrm{SEP}_G(A)$ the set of A-separators of G; and we set

$$\mathrm{disj}_G(A) := \sup\{|\mathscr{D}| : \mathscr{D} \in \mathrm{DISJ}_G(A)\}$$

and

$$\mathrm{sep}_G(A) := \inf\{|(S, T)| : (S, T) \in \mathrm{SEP}_G(A)\}$$

where

$$|(S, T)| := |S| + \sum_{X \in \mathscr{C}_{G(T)}} \lfloor |\mathscr{B}(X, G - S)|/2 \rfloor .$$

When no confusion is likely we will write $\mathrm{DISJ}(A)$, $\mathrm{SEP}(A)$, $\mathrm{disj}(A)$ and $\mathrm{sep}(A)$. Mader proved in [4] that $\mathrm{disj}_G(A) = \mathrm{sep}_G(A)$ for any finite graph G. We will now extend this result to arbitrary infinite graphs.

Theorem 1.2. *Let A be an independent set of vertices of a graph G. Then* $\mathrm{disj}(A) = \mathrm{sep}(A)$.

Proof. We have clearly $\mathrm{disj}(A) \leqslant \mathrm{sep}(A)$. If G is finite, then the converse is Mader's Theorem. Assume that G is infinite. We have two cases:

Case 1: $\mathrm{disj}(A) \geqslant \omega$.

Let $\mathscr{D}$ be an element of $\mathrm{Disj}(A)$ which is maximal with respect to inclusion. Then $(V(\bigcup \mathscr{D}), \emptyset)$ is an A-separator of cardinality $|\mathscr{D}|$ since all A-paths are finite. Hence $\mathrm{disj}(A) = \mathrm{sep}(A)$.

Case 2: $\mathrm{disj}(A) < \omega$.

This is the nontrivial case. Let $\mathscr{D} \in \mathrm{DISJ}(A)$ be of maximum cardinality $n := \mathrm{disj}(A)$. Suppose that $\mathrm{sep}(A) > n$. Let $\mathscr{H}$ be the set of finite induced subgraphs of G containing $\mathscr{D}$, and let $H \in \mathscr{H}$. By Mader's Theorem $\mathrm{disj}_H(A_H) = \mathrm{sep}_H(A_H) = n$, where $A_H = A \cap V(H)$. Let $(S, T) \in \mathrm{SEP}_H(A_H)$ be such that $|(S, T)| = n$. For any $W \in \mathscr{D}$ and any $X \in \mathscr{C}_{H(T)}$ with $E(X) \cap E(W) \neq \emptyset$, we have $|\mathscr{B}(X) \cap V(W)| \geqslant 2$. Thus, since $|(S, T)| = n$, the set of elements of $\mathscr{D}$ meeting S has the same cardinality as S. Hence, since $V(\bigcup \mathscr{D})$ is finite, there is $S \subseteq V(\bigcup \mathscr{D})$ with $|S \cap V(W)| \leqslant 1$ for every $W \in \mathscr{D}$, such that, for any $H \in \mathscr{H}$, there is $T_H \subseteq E(H)$ with $(S, T_H) \in \mathrm{SEP}_H(A_H)$ and $|(S, T_H)| = n$.

For any $H \in \mathscr{H}$ we define the function $\theta_H \in \prod_{e \in E(H)}\{0, 1\}$ so that, if $T_H := \theta_H^{-1}(1)$, then $(S, T_H) \in \mathrm{SEP}_H(A_H)$ with $|(S, T_H)| = n$. W.l.o.g. we can suppose that T_H is such that each component of $G(T_H)$ is an induced subgraph of H, thus of G. By Rado's Selection Principle [9], there is a function $\theta \in \prod_{e \in E(G)}\{0, 1\}$ such that, for any $H \in \mathscr{H}$, there is $H' \in \mathscr{H}$ with $H \subseteq H'$ and $\theta \,|E(H) = \theta_{H'}|\, E(H)$. Let $T := \theta^{-1}(1)$.

Let Y be any finite subgraph of $G \mid T$, and let W be an A-path of G. There is $H \in \mathscr{H}$ such that $Y \cup W \subseteq H$ and $\theta \,|E(Y \cup W \cup \bigcup \mathscr{D}) = \theta_H|\, E(Y \cup W \cup \bigcup \mathscr{D})$. Then, on the one hand, Y is a subgraph of $H \mid T_H$, and this proves that $A \cup S$ does not meet $V(Y)$, thus $\bigcup T$; so T satisfies condition (ii) of Definition 1.1. On the other hand $S \cap V(W) = \emptyset$ implies $T \cap E(W) = T_H \cap E(W) \neq \emptyset$. Thus (S, T) is an A-separator of G.

We will show that $|(S, T)| = n$. Suppose that $|(S, T)| > n$. Then there exist $X_1, \ldots, X_p \in \mathscr{C}_{G(T)}$ with $X_i \neq X_j$ if $i \neq j$, and for $1 \leq i \leq p$ there is a finite subset B_i of $\mathscr{B}(X_i, G - S)$, such that

$$\sum_{1 \leq i \leq p} \lfloor |B_i|/2 \rfloor > n - |S|.$$

For $1 \leq i \leq p$ there is a finite connected induced subgraph Y_i of X_i containing B_i. Let $Y_i' := Y_i \cup \{\langle x, y_x \rangle : x \in B_i\}$ where, for each $x \in B_i$, y_x is a vertex of $G - X_i$ such that $\{x, y_x\} \in E(G) - T$. Since p, as well as each Y_i', is finite, there is $H \in \mathscr{H}$ containing $Y := \bigcup_{1 \leq i \leq p} Y_i'$ such that $\theta_H \,|E(Y \cup \bigcup \mathscr{D}) = \theta|\, E(Y \cup \bigcup \mathscr{D})$. Then for $1 \leq i \leq p$, on the one hand Y_i is a connected subgraph of a component Z_i of $H(T_H)$, and on the other hand $\{x, y_x\} \notin T_H$ for every $x \in B_i$. Thus $B_i \subseteq \mathscr{B}_H(Z_i, H - S)$—where $\mathscr{B}_H$ denotes the boundary in the graph H—since, by the assumption we made, each Y_i' is an induced subgraph of H. Let $\mathscr{Z} := \{Z_1, \ldots, Z_p\}$; notice that we may have $Z_i = Z_j$ for some $i \neq j$. Then

$$\sum_{Z \in \mathscr{Z}} \lfloor |\mathscr{B}_H(Z, H - S)|/2 \rfloor \geq \sum_{1 \leq i \leq p} \lfloor |B_i|/2 \rfloor > n - |S|,$$

which is a contradiction with

$$\sum_{Z \in \mathscr{Z}} \lfloor \mathscr{B}_H(Z, H - S)|/2 \rfloor \leq \sum_{Z \in \mathscr{C}_{H(T_H)}} \lfloor |\mathscr{B}_H(Z, H - S)|/2 \rfloor = n - |S|.$$

hence $|(S, T)| = n$. Therefore $\mathrm{disj}_G(A) = \mathrm{sep}_G(A)$. $\square$

2. A-separators

From now on, $\mathscr{A} := \{A_i : i \in I\}$ will be a set of pairwise disjoint subsets of the end set $\mathfrak{T}(G)$ of some infinite graph G.

Definition 2.1. An ordered pair $(S, T) \in \mathscr{P}(V(G)) \times \mathscr{P}(E(G))$ is an *$\mathscr{A}$-separator* if it has the following properties:

(i) $S \cap \bigcup T = \emptyset$;

(ii) $S \cup \bigcup T \in D(\bigcup \mathscr{A})$;

(iii) for every double ray W consisting of two rays $R_i \in \bigcup A_i$ and $R_j \in \bigcup A_j$ with $\{i, j\} \in [I]^2$, $V(W) \cap S \neq \emptyset$ or $E(W) \cap T \neq \emptyset$.

We will denote by $\mathrm{SEP}(\mathscr{A})$ the set of $\mathscr{A}$-separators and we will set

$$\mathrm{sep}(\mathscr{A}) := \inf\{|S, T)| : (S, T) \in \mathrm{SEP}(\mathscr{A})\}$$

where $|(S, T)|$ is defined as in Definition 1.1.

Remarks 2.2. We have clearly:

(2.2.1) $(S, T) \in \mathrm{SEP}(\mathscr{A})$ if and only if $(S, T) \in \mathrm{SEP}(A_i, A_j)$ for every $\{i, j\} \in [I]^2$.

(2.2.2) If $(S, T) \in \mathrm{SEP}(\mathscr{A})$ then $(S \cup \bigcup T, \emptyset) \in \mathrm{SEP}(\mathscr{A})$.

(2.2.3) $(S, \emptyset) \in \mathrm{SEP}(\mathscr{A})$ if and only if $\mathscr{C}_{G-S}(A_i) \cap \mathscr{C}_{G-S}(A_j) = \emptyset$ for every $\{i, j\} \in [I]^2$.

If $(S, \emptyset)$ is an $\mathscr{A}$-separator, we will say simply that S is an $\mathscr{A}$-separator, and we will write $S \in \mathrm{SEP}(\mathscr{A})$.

Theorem 2.3. $\mathrm{SEP}(\mathscr{A}) \neq \emptyset$ *if and only if* $A_i \cap \overline{\bigcup_{j \neq i} A_j} = \emptyset$ *for every* $i \in I$.

Proof. In the following, if $(X_\xi)_{\xi<\alpha}$ is any sequence we will denote $\bigcup_{\xi<\alpha} X_\xi$ by $X_{(\alpha)}$.

(a) Assume that $\mathscr{A} = \{A_\xi : \xi < \alpha\}$ where α is an ordinal, with $A_\xi \neq A_\eta$ if $\xi \neq \eta$, and suppose that $A_\xi \cap \overline{\bigcup_{\eta \neq \xi} A_\eta} = \emptyset$ for every $\xi < \alpha$. We define by induction a sequence $(S_\xi)_{\xi<\alpha}$ such that, for any $\xi < \alpha$:

(i) $S_{(\xi)} \in D(\bigcup \mathscr{A})$ with $S_{(0)} := \emptyset$.;

(ii) $S_\xi \in D(\bigcup \mathscr{A}) \cap \mathscr{P}(V(G) - S_{(\xi)})$ such that $S_{(\xi+1)} \in \mathrm{SEP}(A_\xi, \bigcup \mathscr{A} - A_\xi)$, and S_ξ is minimal for this property with respect to inclusion.

Suppose that $S_{(\beta)}$ is defined for some $\beta < \alpha$, such that $S_{(\beta)} \in D(\bigcup \mathscr{A})$. Let $G_\beta := G - S_{(\beta)}$, and for each $\xi < \alpha$ let $A_\xi^\beta := \{[R]_{G_\beta} : R$ is a ray of G_β with $[R]_G \in A_\xi\}$. The set $\mathscr{A}^\beta := \{A_\xi^\beta : \xi < \alpha\}$ has the same properties as $\mathscr{A}$, since $S_{(\beta)}$ is $\bigcup \mathscr{A}$-dispersed. By Corollory 2.8 of [7], there is a connected induced subgraph H_β of G_β with $\mathfrak{T}_{H_\beta}(G_\beta) = \overline{\bigcup_{\beta<\xi<\alpha} A_\xi^\beta}$ and such that the boundary of H_β with any component of $G_\beta - H_\beta$ is finite. Let

$$S_\beta := \bigcup_{\tau \in A_\beta^\beta} \mathscr{B}(H_\beta, \mathscr{C}_{G_\beta - H_\beta}(\tau)).$$

Obviously $S_\beta \in D(A_\beta^\beta) \subseteq D(A_\beta)$. Suppose $S_\beta \notin D(\bigcup \mathscr{A} - A_\beta)$. Then $S_\beta \notin D(\bigcup \mathscr{A}^\beta - A_\beta^\beta)$. Thus there is an infinite subset of S_β which is concentrated in some end $\tau \in A_\xi^\beta$ for some $\xi \neq \beta$. But since $\mathscr{B}(H_\beta, \mathscr{C}_{G_\beta - H_\beta}(\tau'))$ is finite for every $\tau' \in A_\beta^\beta$, we have $\tau \in \overline{A_\beta^\beta}$, a contradiction with $A_\xi^\beta \cap \overline{A_\beta^\beta} = \emptyset$. Thus $S_\beta \in D(\mathscr{A})$. Therefore clearly $S_{(\beta+1)} \in \mathrm{SEP}(\mathscr{A}_\beta, \bigcup \mathscr{A} - A_\beta)$, since

$$\mathfrak{T}_{H_\beta}(G_\beta) = \overline{\bigcup_{\beta<\xi<\alpha} A_\xi^\beta},$$

and if $\beta > 0$, $S_{(\beta)} \in \mathrm{SEP}(A_\xi, \bigcup \mathscr{A} - A_\xi)$ for every $\xi < \beta$. Furthermore, by its definition as a boundary, S_β is minimal with respect to inclusion.

It remains to prove that $S_{(\beta)} \in D(\bigcup \mathscr{A})$ for any $\beta < \alpha$. This is clear if β is a successor ordinal, by condition (i) and by the fact that $D(\bigcup \mathscr{A})$ is closer under finite unions. Assume that β is a limit ordinal, and suppose that $S_{(\beta)}$ has an infinite subset which is concentrated in some end θ. For any $\xi < \beta$, by the minimality of S_η for all $\eta < \beta$ (condition (ii)), we have $\mathscr{C}_{G-S_{(\eta)}}(\tau) = \mathscr{C}_{G-S_{(\xi)}}(\tau)$ for every $\tau \in A_\xi$. Thus $\theta \notin A_{(\beta)}$. Suppose $\theta \notin \overline{A_{(\beta)}}$. Then there is a finite subset F of $V(G - \mathscr{C}_{G-S_{(\beta)}}(A_{(\beta)}))$ such that $F \in \mathrm{SEP}(\theta, A_{(\beta)})$. But since, by condition (ii), the

S_ξ's are minimal with respect to inclusion, every element of $S_{(\beta)}$ is adjacent to a vertex of $\mathscr{C}_{G-S_{(\beta)}}(A_{(\beta)})$. Hence, by the definition of F, we must have $\underline{S_{(\beta)}} \cap V(\mathscr{C}_{G-F}(\theta)) = \emptyset$, a contradiction with the definition of θ. Therefore $\theta \in \overline{A_{(\beta)}} - A_{(\beta)}$; and this proves that $\theta \notin \overline{\bigcup \mathscr{A}}$, thus that $S_{(\beta)}$ is $\bigcup \mathscr{A}$-dispersed. Finally $S_{(\alpha)} \in \mathrm{SEP}(\mathscr{A})$.

(b) Conversely suppose that $A_i \cap \overline{\bigcup_{j \neq i} A_j} \neq \emptyset$ for some $i \in I$, and let τ be an element of this intersection. If $S \in D(\bigcup \mathscr{A})$ then $S \cap V(\mathscr{C}_{G-F}(\tau)) = \emptyset$ for some finite subset F of $V(G)$. But $\tau \in \overline{\bigcup_{j \neq i} A_j}$ implies that $\mathscr{C}_{G-F}(\tau) = \mathscr{C}_{G-F}(\tau')$ for some $\tau' \in \bigcup_{j \neq i} A_j$. Hence $S \notin \mathrm{SEP}(A_i, \bigcup_{j \neq i} A_j) \supseteq \mathrm{SEP}(\mathscr{A})$. $\square$

In particular $\mathrm{SEP}([\mathfrak{T}(G)]^1) \neq \emptyset$ if and only if the space $\mathfrak{T}(G)$ is discrete, that is if every end of G is a free end.

2.4. For (S, T), $(S', T') \in \mathrm{SEP}(\mathscr{A})$ we set $(S, T) \leqslant (S', T')$ if and only if $S \subseteq S'$ and $T \subseteq T'$. This relation is obviously an order on $\mathrm{SEP}(\mathscr{A})$. We will show that $\mathscr{A}$ has minimal separators with respect to $\leqslant$.

Proposition 2.5. *The poset* $(\mathrm{SEP}(\mathscr{A}), \geqslant)$ *is inductive.*

Proof. Let $\mathscr{C} := \{(S_k, T_k)\colon k \in K\}$ be a decreasing chain of $\mathrm{SEP}(\mathscr{A})$, and let $S := \bigcap_{k \in K} S_k$ and $T := \bigcap_{k \in K} T_k$. Conditions (i) and (ii) of definition 2.1 are clearly satisfied by (S, T). Let W be a double ray consisting of a ray $W_i \in \bigcup A_i$ and a ray $W_j \in \bigcup A_j$ for some pair $\{i, j\}$ of distinct elements of I. For any $k \in K$, the set $V(W) \cap (S_k \cup \bigcup T_k)$ is finite since $S_k \cup \bigcup T_k$ is $A_i \cup A_j$-dispersed, and non-empty since (S_k, T_k) separates A_i from A_j, with in particular $E(W) \cap T_k \neq \emptyset$ if $V(W) \cap S_k = \emptyset$. Thus, if $V(W) \cap S = \emptyset$, there is $k \in K$ such that $V(W) \cap S_k = \emptyset$, hence $E(W) \cap T_k \neq \emptyset$; and by the finiteness of $E(W) \cap T_k$, this implies that $E(W) \cap T \neq \emptyset$. Therefore (S, T) $\in \mathrm{SEP}(\mathscr{A})$. $\square$

3. *A*-paths

Definition 3.1. Let τ_0 and τ_1 be two distinct ends of G. A *$\tau_0\tau_1$-path* is a finite or infinite path (ray or double ray) W of G such that:

- if W is a double ray then $W = R_0 \cup R_1$ with $R_i \in \tau_i$ for $i = 0, 1$;
- if W is a ray $\langle x_0, x_1, \ldots \rangle$ then $W \in \tau_i$ for some $i \in \{0, 1\}$ and $x_0 \in V_{\tau_{1-i}}$;
- if W is a path $\langle x_0, \ldots, x_n \rangle$ then $x_0 \in V_{\tau_i}$ and $x_n \in V_{\tau_{1-i}}$ for some $i \in \{0, 1\}$.

3.2. For a set $\mathscr{A} := \{A_i\colon i \in I\}$ of pairwise disjoint subsets of $\mathfrak{T}(G)$, we call a $\tau_i\tau_j$-path an *$\mathscr{A}$-path* if $(\tau_i, \tau_j) \in A_i \times A_j$ and $\{i, j\} \in [I]^2$, and we denote by $\mathscr{W}_{\mathscr{A}}$ the set of $\mathscr{A}$-paths, and by $\mathrm{DISJ}(\mathscr{A})$ the set of all sets of pairwise disjoint $\mathscr{A}$-paths. Finally we set

$$\mathrm{disj}(\mathscr{A}) := \sup\{|\mathscr{D}|\colon \mathscr{D} \in \mathrm{DISJ}(\mathscr{A})\}.$$

Theorem 3.3. *If $\mathscr{A}$ is such that $A_i \cap \overline{\bigcup_{j \neq i} A_j} = \emptyset$ for every $i \in I$, then there is a set of pairwise disjoint $\mathscr{A}$-paths of cardinality* $\operatorname{disj}(\mathscr{A})$. *Furthermore, if* $\operatorname{disj}(\mathscr{A})$ *is infinite, then every maximal element of* $\operatorname{DISJ}(\mathscr{A})$ *with respect to inclusion, is of maximum cardinality.*

Proof. This is obvious when $\operatorname{disj}(\mathscr{A})$ is finite, and when it is uncountable since every $\mathscr{A}$-path is countable.

Suppose that $\operatorname{disj}(\mathscr{A}) = \omega$, and let $\mathscr{D}$ be a maximal element of $\operatorname{DISJ}(\mathscr{A})$. Assume $|\mathscr{D}| < \omega$. Then, since $\operatorname{disj}(\mathscr{A}) = \omega$ and since $\mathscr{D}$ is maximal with respect to inclusion, $\mathscr{D}$ has an infinite element containing a ray R such that, for any positive integer n, there is a $\mathscr{D}_n \in \operatorname{DISJ}(\mathscr{A})$ with $|\mathscr{D}_n| \geq n$ and $W \cap R \neq \emptyset$ for all $W \in \mathscr{D}_n$. Thus $[R]_G \in A_i \cap \overline{\bigcup_{j \neq i} A_j}$ for some $i \in I$, a contradiction with the hypothesis. Hence $|\mathscr{D}| = \omega$. □

Using the extension Theorem 1.2 of Mader's Theorem, we will now prove the main result, which will generalize Theorem 3.4 of [8].

Theorem 3.4. *If $\mathscr{A} := \{A_i : i \in I\}$ is such that $A_i \cap \overline{\bigcup_{j \neq i} A_j} = \emptyset$ for every $i \in I$, then* $\operatorname{disj}(\mathscr{A}) = \operatorname{sep}(\mathscr{A})$.

Proof. (a) Let (S, T) be an $\mathscr{A}$-separator, and W be an $\mathscr{A}$-path. Suppose that $V(W) \cap S = \emptyset$ and $|V(W) \cap \mathscr{B}(X, G - S)| < 2$ for any component X of $G(T)$. Then, by condition (iii) of Definition 2.1, W is not a double ray. Thus it has at least an endpoint x which belongs to V_τ for some $\tau \in \bigcup \mathscr{A}$ such that, by the definitions of V_τ and of an $\mathscr{A}$-separator, there is an infinite set of pairwise internally disjoint rays in τ originating at x and meeting S or T. But this implies that $S \cup \bigcup T$ has an infinite subset which is concentrated in τ, a contradiction with condition (ii) of Definition 2.1. Hence $\operatorname{disj}(\mathscr{A}) \leq \operatorname{sep}(\mathscr{A})$.

(b) By Theorem 2.3, (2.2.2) and 0.6, there exists a finitary $\mathscr{A}$-separator S. Let $\Gamma := \{X \in \mathscr{C}_{G-S} : \mathfrak{T}_X(G) \cap \bigcup \mathscr{A} \neq \emptyset\}$, and let $X \in \Gamma$. Since S is finitary, the set $S_X := \mathscr{B}(G \mid S, X)$ is finite; and since S is an $\mathscr{A}$-separator, $A_X := \bigcup \mathscr{A} \cap \mathfrak{T}_X(G) \subseteq A_i$ for some $i \in I$. Let F_X be a finite subset of $S_X \cup V(X)$ such that $S_X \cap V(\mathscr{C}_{G-F_X}(A_X)) = \emptyset$, and which is of minimum cardinality. By Theorem 4.7 of [5] $|F_X|$ is equal to the maximum number of pairwise disjoint (S_X, A_X)-paths (a (S_X, A_X)-path is a path or a ray W originating in S_X such that, if W is a ray then $[W]_G \in A_X$, and otherwise $V(W) \cap V_{A_X} \neq \emptyset$, and which is minimal with this property with respect to inclusion).

For $i \in I$, we denote by Γ_i the subset of Γ such that $A_i = \bigcup_{X \in \Gamma_i} A_X$, and by a_i a vertex so that $a_i \notin V(G)$, and $a_i \neq a_j$ if $i \neq j$. Finally we define the graphs:

$$G'_{\mathscr{A}} := G - \bigcup_{X \in \Gamma} \mathscr{C}_{G-F_X}(A_X)$$

and

$$G_{\mathscr{A}} := \bigcup_{1 \leqslant i \leqslant p} \left\{ \langle a_i, x \rangle : x \in \bigcup_{X \in \Gamma_i} F_X \right\} \cup G'_A.$$

Notice that, by the definitions of S and of the F_X's $V_{\tau_i} \cap V_{\tau_j} \subseteq V(G_{\mathscr{A}})$ for all $\{i, j\} \in [I]^2$ and $(\tau_i, \tau_j) \in A_i \times A_j$.

Then, still by the definition of the F_X's, for any family of pairwise disjoint $\mathscr{A}$-paths $\mathscr{D}$ there is a family of pairwise internally disjoint $\{a_i: i \in I\}$-paths $\mathscr{D}'$ such that

$$\{W \cap G'_{\mathscr{A}}: W \in \mathscr{D}\} = \{W \cap G'_{\mathscr{A}}: W \in \mathscr{D}'\};$$

and conversely. Therefore $\mathrm{disj}(\mathscr{A}) = \mathrm{disj}(\{a_i: i \in I\})$. On the other hand

$$\mathrm{SEP}_{G_{\mathscr{A}}}(\{a_i: i \in I\}) = \{(S, T) \in \mathrm{SEP}(\mathscr{A}): S \cup T \subseteq V(G'_{\mathscr{A}})\}.$$

Consequently, by the extension Theorem 1.2 of Mader's Theorem,

$$\mathrm{sep}(\mathscr{A}) \geqslant \mathrm{disj}(\mathscr{A}) = \mathrm{disj}_{G_{\mathscr{A}}}(\{a_i: i \in I\}) = \mathrm{sep}_{G_{\mathscr{A}}}(\{a_i: i \in I\}) \geqslant \mathrm{sep}(\mathscr{A}).$$

Hence $\mathrm{disj}(\mathscr{A}) = \mathrm{sep}(\mathscr{A})$. $\square$

When $\mathscr{A}$ has only two elements, Theorem 3.3 of [8] gives the following precision: *If* $|\mathscr{A}| = 2$ *then there is* $(S, \emptyset) \in \mathrm{SEP}(\mathscr{A})$ *such that* $|S| = \mathrm{sep}(\mathscr{A}) = \mathrm{disj}(\mathscr{A})$.

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